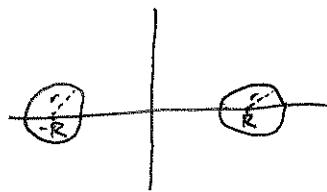


Torus.

Cross - section

Solutions, Q5.



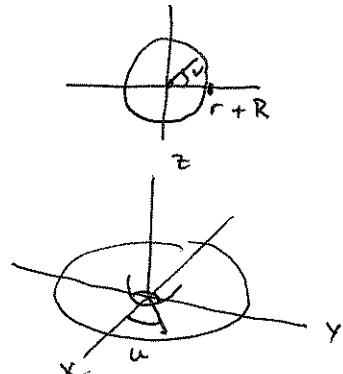
Parametrization.

Let u = angle in large circle v = angle in small circle

Then $z = r \sin v$

$x = (R + r \cos v) \cos u$

$y = (R + r \cos v) \sin u$



$$\text{Check: } z^2 + (x^2 + y^2)^2 = (r \sin v)^2 + ((R + r \cos v) \cos u)^2 + ((R + r \cos v) \sin u)^2 = r^2 \sin^2 v + r^2 \cos^2 v = r^2 \checkmark.$$

The surface area:

$$= \int_0^{2\pi} \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \, du \, dv \quad \text{where } \vec{r} = \langle (R + r \cos v) \cos u, (R + r \cos v) \sin u, r \sin v \rangle.$$

$$\vec{r}_u = \langle - (R + r \cos v) \sin u, (R + r \cos v) \cos u, 0 \rangle$$

$$\vec{r}_v = \langle - r \sin v \cos u, - r \sin v \sin u, r \cos v \rangle$$

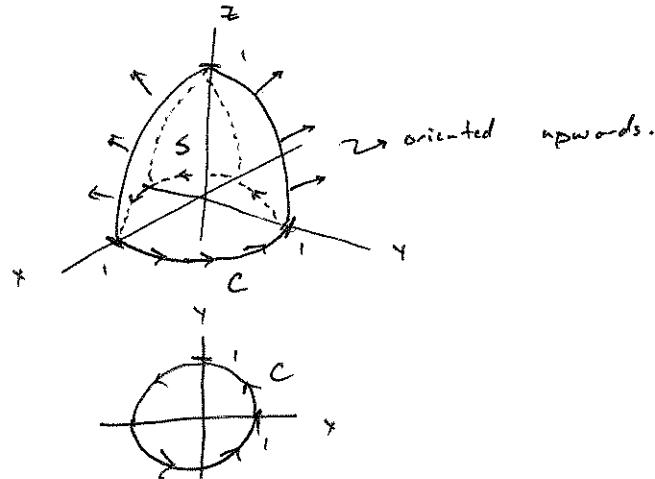
$$\begin{aligned} \vec{r}_u \times \vec{r}_v &= \langle (R + r \cos v) r \cos v \cos u, (R + r \cos v) r \cos v \sin u, \\ &\quad (R + r \cos v) r \sin^2 u \sin v + (R + r \cos v) r \cos^2 u \sin v \rangle \\ &= r(R + r \cos v) \langle \cos v \cos u, \cos v \sin u, \sin v \rangle \end{aligned}$$

Hence $|\vec{r}_v \times \vec{r}_v| = r(R + r \cos v)$ after using $\cos^2 v + \sin^2 v = 1$ twice.

$$\begin{aligned} \text{Therefore, } SA &= \int_0^{2\pi} \int_0^{2\pi} r(R + r \cos v) d\omega dv \\ &= r \int_0^{2\pi} Rv + r \sin v \Big|_{v=0}^{v=2\pi} dv \\ &= rR \int_0^{2\pi} 2\pi dv \\ &= 4\pi^2 rR \end{aligned}$$

as desired. □

2. $z = 1 - x^2 - y^2$: paraboloid pointing down.



The boundary curve is
a circle, $x^2 + y^2 = 1$.

The orientation is CCW.

First, the line integral: Parameterize as $\vec{r} = \langle \cos t, \sin t, 0 \rangle$

so that $d\vec{r} = \langle -\sin t, \cos t, 0 \rangle dt$, $0 \leq t \leq 2\pi$.

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \langle \cos^2 t, \sin^2 t, 0^2 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt \\ &= \int_0^{2\pi} -\sin t \cos^2 t + \cos t \sin^2 t dt \\ &= \frac{1}{3} (\cos^3 t + \sin^3 t) \Big|_0^{2\pi} = 0. \end{aligned}$$

Now the surface integral:

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}.$$

So we don't need to parameterize:

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_S \vec{0} \cdot d\vec{S} = \iint_S \underbrace{\vec{0} \cdot \vec{n}}_0 dS = 0.$$

So Stokes' Theorem holds in this case. ✓